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On effective superpotentials and compactification to three dimensions

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Abstract: We study four dimensional $\mathcal{N} = 2$ SO/SP supersymmetric gauge theory on $R^3 \times S^1$ deformed by a tree level superpotential. We will show that the exact superpotential can be obtained by making use of the Lax matrix of the corresponding integrable model which is the periodic Toda lattice. The connection between vacua of $\mathrm{SO}(2N)$ and $\mathrm{SO}(2kN-2k+2)$ can also be seen in this framework. Similar analysis can also be applied for $\mathrm{SO}(2N+1)$ and $\mathrm{SP}(2N)$.

Keywords: Nonperturbative Effects, Supersymmetry and Duality, Supersymmetric Effective Theories.

Contents

1. Introduction

It is believed that supersymmetric field theories have a rich structure which makes it possible to study some nonperturbative dynamics of the theory such as gaugino condensation or confinement and chiral symmetry braking exactly (for a review, see e.g [1]). This is mainly because one can find the exact form of the effective superpotential and thereby the structure of supersymmetric vacuum.

Recently an interesting step toward describing the vacuum structure, as well as low energy coupling for a wide class of $\mathcal{N} = 1$ supersymmetric gauge theories was given by Dijkgraaf and Vafa [2]. Motivated by earlier works [3]–[7] these authors have conjectured that the exact superpotential and gauge coupling for these $\mathcal{N} = 1$ theories can be obtained using perturbative computations in a related matrix model. Given an $\mathcal{N} = 1$ SYM with a tree level superpotential, the potential of the corresponding matrix model is given in terms of the gauge theory tree level superpotential. This conjecture has been verified using superspace perturbation formalism [8] or anomalies [9].

More recently, in another attempt to study exact results in three dimensions, the authors of [10] have considered $\mathcal{N}=2$ U(N) supersymmetric gauge theory deformed by a tree level superpotential on $R^3\times S^1$.[1] In fact, based on earlier works [11], they have observed that the vacuum structure of the theory can be described using an integrable model that underlies the four dimensional theory which in their case is the periodic Toda lattice based on the root system of the affine Lie algebra $A^{(1)}_{N-1}$.[2] More precisely the authors of [10] have conjectured that if the classical superpotential is Tr$W(\phi)$, then the quantum superpotential is given by Tr$W(M)$, where M is the Lax matrix of the corresponding integrable model. They have also shown that the structure of the supersymmetric vacuum can also be studied in this framework, in particular it was shown that the vacuum structure of SU(N) gauge theory can be lifted to that of SU(KN) gauge theory.

It is the aim of this article to further investigate this proposal. In fact we will see that the proposal also works for $\mathcal{N}=2$ SYM theory with SO / SP gauge groups. In this case the integrable model is also given in terms of the periodic Toda lattice, though in the case of nonsimply laced groups one needs to work with the dual gauge group where the short roots and long roots are exchanged [18].

The paper is organized as follows. In section 2 we shall review some field theory aspects of $\mathcal{N}=2$ SO($2N$) supersymmetric gauge theory on $R^3\times S^1$ and its reduction to three dimensions as well. In section 3 we will demonstrate the way in which the effective superpotential can be obtained for $\mathcal{N}=2$ SO($2N$) supersymmetric gauge theory on $R^3\times S^1$ deformed by a tree level superpotential W. This has been done by making use of an integrable system. In fact we will see that the Lax matrix of the corresponding integrable model plays an essential role. In section 4 we shall show that the lifting of supersymmetric vacua of SO($2N$) to SO($2KN-2K+2$) can also be understood in the context of integrable model. In the section 5 the same scenario has been applied for SO($2N+1$) and SP($2N$) gauge groups. The last section is devoted to conclusion and comments.

2. Field theory description

2.1 Three dimensional $\mathcal{N}=2$ SO($2N$) SYM theory

In this section we shall consider $\mathcal{N}=2$ SO($2N$) supersymmetric gauge theory in three dimensions. Having a gauge theory one should deal with the vector multiplet of three dimensional SUSY theory.

The vector multiplet V, of three dimensional $\mathcal{N}=2$ supersymmetric theory contains the gauge field, an adjoint scalar and two real fermion gauginos which can be combined into a complex fermion. The chiral/anti-chiral field strengths are defined by $W_\alpha=-\frac{1}{4}\bar{D}^2e^{-V}D_\alpha e^V$ and $\bar{W}_\alpha=-\frac{1}{4}D^2e^{-V}\bar{D}_\alpha e^V$, and the kinetic term of the classical

[1]The compactification of the $\mathcal{N}=2$ SYM theory to three dimensions was considered in [12]. For further discussions see for example [13]–[16].

[2]The relation between $\mathcal{N}=2$ SYM theories and integrable system was discussed in several papers including [17]–[24]. For recent discussion in this direction and its relation with Dijkgraaf-Vafa conjecture see also [25].

action is given by

$$\frac{1}{g_3^2}\int d^3x d^2\theta \, \mathrm{Tr}(W^\alpha W_\alpha) + \mathrm{h.c.}\,. \tag{2.1}$$

The theory has a Coulomb branch where the real scalar ϕ, which is taken in the Cartan subalgebra of the gauge group, gets an expectation value. In a generic point of the moduli space the gauge group is broken to the Cartan subgroup where the gauge group is $\mathrm{U}(1)^N$, while at the boundaries of the Weyl chamber there is classically enhanced gauge symmetry.

The $\mathrm{U}(1)^N$ gauge fields, in the bulk of Coulomb branch, can be dualized to scalars, $F^a_{\mu\nu} = \epsilon_{\mu\nu\rho}\partial^\rho\sigma^a$, $a = 1, \ldots, N$. The σ^a which parameterize the Cartan torus of the gauge group, can be combined with the ϕ^a into chiral superfield Φ^a whose scalar components are $\phi^a + i\sigma^a$. One can also think about this theory as compactification of four dimensional $\mathcal{N} = 1$ gauge theory in which ϕ can be thought of as the component of the gauge field in the reduced direction.

For three dimensional $\mathcal{N} = 2$ gauge theory one expects to get a superpotential. The nonperturbative contribution is due to three dimensional instantons which are monopoles in the four dimensional theory. These three dimensional instantons are associated with π_2 and since $\pi_2(\mathrm{SO}(2N)) = 0$ there can only be instantons in the Coulomb branch where the gauge group is broken to $\mathrm{U}(1)^N$ and one has $\pi_2(\mathrm{SO}(2N)/\,\mathrm{U}(1)^N) = Z^N$. Therefore there are N independent fundamental instantons associated with the simple roots of the gauge group. Their contribution to the nonperturbative superpotential is given by

$$W = \sum_{i=1}^{N-1} e^{\frac{\phi_i - \phi_{i+1}}{g_3^2}} + e^{\frac{\phi_{N-1}+\phi_N}{g_3^2}}\,. \tag{2.2}$$

2.2 Four dimensional SYM theory on $R^3 \times S^1$

If we think about this three dimensional theory as a theory which comes from reduction of a four dimensional $\mathcal{N} = 1$ gauge theory on a circle of radius R with $g_3^{-2} = Rg_4^{-2}$, the superpotential develops an R dependent term [12] which in the case of $\mathrm{SO}(2N)$ gauge theory is given by [13]

$$W = \sum_{i=1}^{N-1} e^{\frac{\phi_i - \phi_{i+1}}{g_3^2}} + e^{\frac{\phi_{N-1}+\phi_N}{g_3^2}} + \gamma e^{(-\frac{\phi_1+\phi_2}{g_3^2})}\,. \tag{2.3}$$

where $\gamma = e^{-1/Rg_3^2}$. From group theory point of view the last term corresponds to the extra node one can add to the Dynkin diagram to make an affine Dynkin diagram. The three dimensional theory is obtained in the limit of $R \to 0$ while the four dimensional theory is recovered in large R limit.

The theory could also have a mass term for real scalar fields ϕ_i's and therefore the whole superpotential reads

$$W = \frac{1}{2}m\sum_{i=1}^{N}\phi_i^2 + \sum_{i=1}^{N-1} e^{\frac{\phi_i - \phi_{i+1}}{g_3^2}} + e^{\frac{\phi_{N-1}+\phi_N}{g_3^2}} + \gamma e^{(-\frac{\phi_1+\phi_2}{g_3^2})}\,, \tag{2.4}$$

which under a field redefinition can be recast to the following form

$$W = \frac{1}{2}m\sum_{i=1}^{N}\phi_i^2 + m\sum_{i=0}^{N} y_0\,, \tag{2.5}$$

where

$$my_0 = \gamma e^{(-\frac{\phi_1+\phi_2}{g_3^2})}\,, \qquad my_N = e^{\frac{\phi_{N-1}+\phi_N}{g_3^2}}\,, \qquad my_i = e^{\frac{\phi_i-\phi_{i+1}}{g_3^2}} \quad \text{for } i=1,\ldots,N-1\,. \tag{2.6}$$

We note, however, that there is a constraint on the variables y_i which is

$$\prod_{i=0}^{N} y_i \prod_{j=2}^{N-2} y_j = \gamma m^{-(2N-2)} = \Lambda^{4N-4}\,, \tag{2.7}$$

with Λ being the dynamical scale. Therefore one needs to impose this constraint in the superpotential using a Lagrange multiplier L, as

$$W = \frac{1}{2}m\sum_{i=1}^{N}\phi_i^2 + m\sum_{i=0}^{N} y_0 + L\log\left(\frac{\Lambda^{4N-4}}{\prod_{i=0}^{N} y_i \prod_{j=2}^{N-2} y_j}\right). \tag{2.8}$$

We could also consider the theory with a general classical superpotential given by a holomorphic function $W(\Phi)$ on $R^3\times S^1$. Then the question would be how one can find the exact superpotential in this case. In the next section we will study this problem using the fact that this model is related to an integrable model.

3. Quantum superpotential

In this section we would like to study the exact superpotential of $\mathcal{N}=1$ SO($2N$) SYM theory on $R^3\times S^1$ which can be obtained from $\mathcal{N}=2$ Seiberg-Witten model deformed by a classical superpotential $\int d^4x d^2\theta W(\Phi)$. To do this we note that the Seiberg-Witten model is related to an integrable system which is the periodic Toda lattice [17, 18, 19].

The periodic Toda lattice associated to SO($2N$) gauge group is given by the following hamiltonian [26]

$$H = \frac{1}{2}\sum_{i=1}^{N} p_N^2 + \sum_{i=0}^{N} V_i\,, \tag{3.1}$$

where

$$\begin{aligned} V_i &= \Lambda^2 e^{q_i-q_{i+1}} \quad \text{for } i=1,\ldots,N-1\,,\\ V_N &= \Lambda^2 e^{q_{N-1}+q_N}\,, \qquad V_0 = \Lambda^2 e^{-(q_1+q_2)}\,, \end{aligned} \tag{3.2}$$

here q_i are coordinates and p_i are their corresponding momenta, and Λ^2 is a parameter which will play the role of the dynamical scale in the gauge theory side. This model is an integrable model which means that there exists a Lax pair given by two matrices $M_{2N\times 2N}$

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and $A_{2N\times 2N}$ that are functions of coordinates and momenta such that evolution of the theory can be described by the Lax equation

$$\frac{\partial M}{\partial t} = [M, A]\,. \tag{3.3}$$

The Lax matrix M, for SO($2N$) group is given by [27]

$$M = \begin{pmatrix}
p_1 & V_1 & & & & & & & -z & 0 \\
1 & p_2 & V_2 & & & & & & & z \\
0 & 1 & & & & & & & & \\
 & & & & V_{N-1} & -V_N & & & & \\
 & & & 1 & p_n & 0 & V_N & & & \\
 & & & -1 & 0 & -p_N & -V_{N-1} & & & \\
 & & & & 1 & -1 & -p_{N-1} & -V_{N-2} & & \\
 & & & & & & -1 & & & \\
-\frac{V_0}{z} & & & & & & & & & -V_1 \\
0 & \frac{V_0}{z} & & & & & & & -1 & -p_1
\end{pmatrix}.$$

We note also that there is a constraint on V_i's, namely $\prod_{i=0}^{N} V_i \prod_{i=2}^{N-2} V_i = \Lambda^{4N-4}$. To make a connection with our discussion in the previous section it is useful to make a change of variable in which $p_i = \phi_i$ and $V_i = y_i$. In particular we note that

$$\mathrm{Tr}(M^2) = 4\left(\frac{1}{2}\sum_{i=1}^{N}\phi_i^2 + \sum_{j=0}^{N} y_i\right) \tag{3.4}$$

which is the same as (2.8) up to a factor of two which is because of Z_2 symmetry of the root system of SO($2N$). In fact this is a special case of the proposal made in [10], namely the quantum superpotential can be obtained from the classical one by replacing ϕ with the Lax matrix M

$$\int dx^4 d^2\theta\; W(\Phi) \to \int dx^4 d^2\theta\; W(M)\,. \tag{3.5}$$

Of course to compare this with the gauge theory result one needs to identify the integrable system parameters (p_i, q_i) with the gauge theory fields as what we have done in equation (3.4). Finally we note that the spectral parameter z, does not appear in the quantum potential as long as we consider the classical superpotential with powers less than $2N-2$. For a potential with a term of $\mathrm{Tr}(\phi^{2N-2})$ one finds a constant z dependent term in the form of $(4N-4)(z + \frac{\Lambda^{4N-4}}{z})$ which, following [10], we will simply drop!

3.1 An example

To see how the proposal works for the SO($2N$) gauge theory, let us consider the theory with the gauge group SO(8) with tree level superpotential

$$W = \frac{g_2}{2}\,\mathrm{Tr}(\phi^2) + \frac{g_4}{4}\,\mathrm{Tr}(\phi^4)\,. \tag{3.6}$$

The corresponding Lax matrix for gauge group SO(8) is given by

$$M = \begin{pmatrix} \phi_1 & y_1 & 0 & 0 & 0 & 0 & -z & 0 \\ 1 & \phi_2 & y_2 & 0 & 0 & 0 & 0 & z \\ 0 & 1 & \phi_3 & y_3 & -y_4 & 0 & 0 & 0 \\ 0 & 0 & 1 & \phi_4 & 0 & y_4 & 0 & 0 \\ 0 & 0 & -1 & 0 & -\phi_4 & -y_3 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & -\phi_3 & -y_2 & 0 \\ -\frac{y_0}{z} & 0 & 0 & 0 & 0 & -1 & -\phi_2 & -y_1 \\ 0 & \frac{y_0}{z} & 0 & 0 & 0 & 0 & -1 & -\phi_1 \end{pmatrix}, \tag{3.7}$$

The Seiberg-Witten curve is obtained from the spectral curve given by $\det(x\mathbf{1} - M) = 0$ which reads[3]

$$x^2\left(z + \frac{\Lambda^{12}}{z}\right) - \frac{1}{4}(x^8 - u_2x^6 + \cdots + u_8) = 0\,, \tag{3.8}$$

where $y_2\prod_{i=0}^4 y_i = \Lambda^{12}$.

Following [10] the effective superpotential reads

$$W_{\text{eff}} = \frac{g_2}{2}\,\mathrm{Tr}(M^2) + \frac{g_4}{4}\,\mathrm{Tr}(M^4) + 2L\log\left(\frac{\Lambda^{12}}{y_2^2\prod_{i=0}^4 y_i}\right). \tag{3.9}$$

Here we have also imposed the constraint on y_i's using a Lagrange multiplier L. The equations of motion for ϕ_i's are given by

$$\begin{aligned} W'(\phi_1) + g_4(y_1\phi_2 - y_0\phi_2 + 2y_1\phi_1 + 2y_0\phi_1) &= 0\,, \\ W'(\phi_2) + g_4(y_2\phi_3 + y_1\phi_1 - y_0\phi_1 + 2y_1\phi_2 + 2y_2\phi_2 + 2y_0\phi_2) &= 0\,, \\ W'(\phi_3) + g_4(y_3\phi_4 - y_4\phi_4 + y_2\phi_2 + 2y_3\phi_3 + 2y_4\phi_3 + 2y_2\phi_3) &= 0\,, \\ W'(\phi_4) + g_4(y_3\phi_3 - y_4\phi_3 + 2y_3\phi_4 + 2y_4\phi_4) &= 0\,, \end{aligned} \tag{3.10}$$

where $W'(\phi_i) = g_2\phi_i + g_4\phi_i^3$, while the equations of motion for y_i's read

$$\begin{aligned} g_2 + g_4(y_0 + y_2 + 3y_1 + \phi_2^2 + \phi_1^2 - \phi_1\phi_2) &= Ly_0^{-1}\,, \\ g_2 + g_4(y_1 + y_2 + 3y_0 + \phi_2^2 + \phi_1^2 + \phi_1\phi_2) &= Ly_1^{-1}\,, \\ g_2 + g_4(y_2 + y_1 + y_3 + y_4 + y_0 + \phi_3^2 + \phi_2^2 + \phi_2\phi_3) &= 2Ly_2^{-1}\,, \\ g_2 + g_4(y_3 + y_2 + 3y_4 + \phi_3^2 + \phi_4^2 + \phi_3\phi_4) &= Ly_3^{-1}\,, \\ g_2 + g_4(y_4 + y_2 + 3y_3 + \phi_3^2 + \phi_4^2 - \phi_3\phi_4) &= Ly_4^{-1}\,. \end{aligned} \tag{3.11}$$

Moreover the equation of motion of L gives $y_2\prod_{i=1}^4 y_i = \Lambda^{12}$.

Any solution of these equations would lead to a supersymmetric vacuum of the theory. In particular one could consider solutions with $\phi_1 = \phi_4 = 0$, $\phi_2 = \phi_3 = \phi$ and $y_i = y$ for $i \neq 2$. In this case we are left with the following equations for ϕ, y and y_2

$$W'(\phi) + g_4(4y + 3y_2)\phi = 0\;,$$

[3]The Seiberg-Witten curve for SO($2N$) SYM theory was first obtained in [28].

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$$\begin{aligned} g_2 + g_4(y_2 + 4y + \phi^2) &= \frac{L}{y}\,, \\ g_2 + g_4(y_2 + 4y + 3\phi^2) &= \frac{2L}{y_2}\,. \end{aligned} \tag{3.12}$$

Let us first study a situation where $\phi = 0$. For this case one finds

$$\phi_i = 0\,, \qquad y_2 = 2y\,, \qquad y_i = y \quad \text{for } i = 0,1,3,4\,, \qquad L = y(g_2 + 6g_4 y)\,. \tag{3.13}$$

where $y = (1/2^{1/3})\Lambda^2$. This solution corresponds to the case where classically the gauge group remains unbroken and thus we get maximally confining vacuum.

Plugging this solution into the effective superpotential (3.9) one finds

$$W_{\text{eff}} = 6\left(2^{2/3} g_2 \Lambda^2 + 3\, 2^{1/3} g_4 \Lambda^4\right). \tag{3.14}$$

Let us now assume that $\phi \neq 0$ then the solution is give by

$$\phi^2 = -\left(\frac{g_2}{4g_4} + \frac{5}{2}y\right), \qquad y_2 = -\left(\frac{g_2}{4g_4} + \frac{1}{2}y\right), \qquad y_i = 2y \quad \text{for } i = 0,1,3,4\,. \tag{3.15}$$

where y is obtained from the following equation

$$\frac{1}{2}y^3 + \frac{g_2}{4g_4}y^2 + \Lambda^6 = 0\,, \tag{3.16}$$

which can be solved in power series of Λ. The result is

$$y = -\frac{g_2}{2g_4} - 8\frac{g_4^2}{g_2^2}\Lambda^6 + 256\frac{g_4^5}{g_2^5}\Lambda^{12} - 14336\frac{g_4^8}{g_2^8}\Lambda^{18} + 983040\,\frac{g_4^{11}}{g_2^{11}}\Lambda^{24} + \mathcal{O}(\Lambda^{28})\,. \tag{3.17}$$

Therefore the effective superpotential (3.9) for this solution reads

$$W_{\text{eff}} = \frac{g_2^2}{g_4}\left(-1 + 16\,\frac{g_4^3}{g_2^3}\,\Lambda^6 - 128\,\frac{g_4^6}{g_6^3}\,\Lambda^{12} + 4096\,\frac{g_4^9}{g_2^9}\,\Lambda^{18} - 196608\,\frac{g_4^{12}}{g_2^{12}}\,\Lambda^{24}\right) + \mathcal{O}(\Lambda^{28})\,. \tag{3.18}$$

From the solution we found, one can see that this vacuum corresponds to the case where the gauge group is classically broken as $\mathrm{SO}(8) \to \mathrm{SO}(4) \times \mathrm{U}(2)$.

These results should be compared with the 4-dimensional field theory. To do this we note that the 4-dimensional field theory result could be obtained using factorization of the corresponding Seiberg-Witten curve. In general the factorization problem is hard to solve, but for the maximally confining vacuum there is a general solution given by Chebyshev polynomials. For $\mathrm{SO}(N)$ gauge group the solution is [31]

$$\langle u_{2p}\rangle = \frac{\tilde{h}}{2p}\binom{2p}{p}\Lambda^{2p}\,. \tag{3.19}$$

where $\tilde{h}$ is dual Coxeter number of the gauge group. Therefore in our model the effective superpotential reads

$$W_{\text{eff}} = g_2\langle u_2\rangle + g_4\langle u_4\rangle = 6\left(g_2\Lambda^2 + \frac{3}{2}g_4\Lambda^4\right) \tag{3.20}$$

which is the same as (3.14) if we rescale $\Lambda \to 2^{1/3}\Lambda$. This rescaling can also be understood from the spectral curve we have found using our notation for Lax matrix M. In fact in comparison with the spectral curve in [18] we have an extra factor of $\frac{1}{4}$ which can be absorbed by a redefinition of z and Λ. Actually one needs to rescale $z \to z/4$ and $\Lambda \to \Lambda/4^{(4/4N-4)}$ to make our notation as that in [18]. Therefore in the case of SO(8) one gets a rescaling factor of $2^{1/3}$.

In the case where the gauge symmetry is broken as $\mathrm{SO}(8) \to \mathrm{SO}(4) \times \mathrm{U}(2)$, the factorization of Seiberg-Witten can also be worked out. In fact this model has been studied in [32] where the effective superpotential was given in a power series of Λ as following

$$W_{\text{eff}} = \frac{g_2^2}{g_4}\left(-1 + 4\,\frac{g_4^3}{g_2^3}\,\Lambda^6 - 8\,\frac{g_4^6}{g_6^3}\,\Lambda^{12} + 64\,\frac{g_4^9}{g_2^9}\,\Lambda^{18} - 768\,\frac{g_4^{12}}{g_2^{12}}\,\Lambda^{24}\right) + \mathcal{O}(\Lambda^{28})\,, \tag{3.21}$$

which is the same as (3.18) upon the rescaling of $\Lambda \to 2^{1/3}\Lambda$, as expected.

Note that if we had considered a $\frac{g_6}{6}\mathrm{Tr}(\phi^6)$ term in the classical superpotential one would have got the following quantum superpotential

$$W = 6\left(2^{2/3}g_2\Lambda^2 + 3\,2^{1/3}g_4\Lambda^4 + \frac{40}{3}g_6\Lambda^6\right) + 4g_6\left(z + \frac{\Lambda^{12}}{z}\right), \tag{3.22}$$

which has a z dependent term. We note also that the Λ^6 term is indeed in agreement with the field theory result coming from (3.19). Therefore, as we mentioned before, it seems that the z dependent term has no contribution and should be dropped [10], though we do not have any strong physical reason for it.

4. SO$(2KN - 2K + 2)$ gauge theory from SO$(2N)$ theory

It has been shown [32] that a solution for the massless monopole constraints of the $\mathrm{SO}\times(2KN - 2K + 2)$ can be obtained from that of SO(N) via Chebyshev polynomials. Since the compactified theory is closely related to an integrable system one might wonder if such a structure could also be seen in this framework. In this section we shall examine this relation using their corresponding integrable systems. In fact we will show that this information is indeed encoded in the structure of the Lax matrix of both theories. Actually our discussion in this section is parallel to that in [10] where the lifting from SU(N) to SU(KN) was studied.

Consider the following $2KN \times 2KN$ matrix made out of the Lax matrix of the SO$(2N)$ periodic Toda lattice

$$M_{2KN} = \begin{pmatrix}
\phi_1 & y_1 & & & & & & & & & -z & 0 \\
1 & & & & & & & & & & 0 & z \\
 & & & & & & & & & & & \\
 & & & y_{N-1} & -y_N & & & & & & & \\
 & & 1 & \phi_N & 0 & y_N & & & & & & \\
 & & -1 & 0 & -\phi_N & -y_{N-1} & & & & & & \\
 & & & 1 & -1 & & & & & & & \\
 & & & & & & & & & & & \\
 & & & & & & & -y_1 & -y_0 & & & \\
 & & & & & & -1 & -\phi_1 & 0 & y_0 & & \\
 & & & & & & -1 & 0 & \phi_1 & y_1 & & \\
 & & & & & & & 1 & 1 & & & \\
 & & & & & & & & & & & \\
\frac{-y_0}{z} & 0 & & & & & & & & & & -y_1 \\
0 & \frac{y_0}{z} & & & & & & & & & -1 & -\phi_1
\end{pmatrix}.$$

Following [10] let us define a matrix G as

$$G = \mathrm{diag}\left(1, z^{1/2KN}, z^{2/2KN}, \ldots, z^{(2KN-1)/2KN}\right) \tag{4.1}$$

and map the matrix M into $\tilde{M}$ using a similarity transformation $\tilde{M} = GMG^{-1}$. It can be seen that the obtained matrix $\tilde{M}$ satisfies $S\tilde{M}S^{-1} = \tilde{M}$, where

$$S = \begin{pmatrix} 0 & I_{2(K-1)N} \\ I_{2N} & \end{pmatrix} \tag{4.2}$$

is a matrix that generates a cyclic permutation of order $2N$ on $2KN$ elements. For the later use we note that the eigenvectors of S are $2KN$-dimensional space which is a direct sum of K $2N$-dimensional subspaces labeled by a different $K-th$ root of unity $e^{\frac{2r\pi i}{2N}}$. Thus a basis for eigenvectors of S might be recast to $v_r^j = v^j(1, e^{\frac{2r\pi i}{2N}}, e^{\frac{4r\pi i}{2N}}, \ldots, e^{\frac{(K-1)r\pi i}{2N}})$ for $j = 1, 2, \ldots, 2N$.

Now the task is to show that the M_{2KN} is somehow *related* to the proper Lax matrix of the SO$(2KN - 2K + 2)$ gauge group. The invariance under cyclic permutation ensures that the equations resulting from $\tilde{M}_{2KM}$ collapse to those resulting from M_{2N}. In order to find the spectral curve resulting from M_{2KN}, we note that the matrix $\tilde{M}$ can explicitly be written as the following

$$\tilde{M} = \begin{pmatrix}
A & B & 0 & . & . & . & C \\
C & A & B & 0 & . & . & 0 \\
0 & C & A & B & . & . & . \\
. & . & . & . & . & . & B \\
B & 0 & 0 & 0 & . & C & A
\end{pmatrix} \tag{4.3}$$

where A, B and C are $2N \times 2N$ matrices given by

$$
A = \begin{pmatrix}
\phi_1 & \frac{y_1}{z^{1/2KN}} & & & & & 0 \\
z^{1/2KN} & & & & & & \\
& & & \frac{y_{N-1}}{z^{1/2KN}} & \frac{-y_N}{z^{1/KN}} & & \\
& & & \phi_N & 0 & \frac{y_N}{z^{1/KN}} & \\
& & -z^{1/KN} & 0 & -\phi_N & \frac{-y_{N-1}}{z^{1/2KN}} & \\
& & & z^{1/KN} & & & \\
& & & & & & \frac{-y_1}{z^{1/2KN}} \\
0 & & & & & & -\phi_1
\end{pmatrix},
$$

$$
B = y_0 z^{-1/KN} \begin{pmatrix} 0 & . & . & 0 \\ . & . & . & . \\ -1 & 0 & . & 0 \\ 0 & 1 & . & 0 \end{pmatrix}, \quad C = z^{1/KN} \begin{pmatrix} 0 & . & -1 & 0 \\ 0 & . & 0 & 1 \\ . & . & . & . \\ 0 & . & . & 0 \end{pmatrix}. \tag{4.4}
$$

Since the matrix $\tilde{M}$ commutes with S one can use the basis of eigenvectors of S to diagonalize it and thereby to compute its determinant in terms of A, B and C matrices as the following [10]

$$
\det\left(\tilde{M}\right) = \prod_{t=1}^{K} \det\left(A + e^{\frac{2i\pi t}{K}} B + e^{\frac{-2i\pi t}{K}} C\right). \tag{4.5}
$$

By making use of this relation, let us now evaluate the spectral curve coming from the *auxiliary Lax* matrix $\tilde{M}$. From the structure of matrix $\tilde{M}$ one finds

$$
\det(x\mathbf{1} - \tilde{M}) = \prod_{t=1}^{K} \left[P_{2N}(x) - x^2 \left(e^{\frac{-2t\pi i}{K}} z^{\frac{1}{t}} + \eta^2 e^{\frac{2t\pi i}{K}} z^{-\frac{1}{t}}\right)\right] + x^{2K}\left(z + \frac{\lambda^{4KN-4K}}{z}\right) = 0\,, \tag{4.6}
$$

where η is a $2K$-th root of unity. Note that the x-dependent factor in front of the z term comes from the fact that the matrix $\tilde{M}$ is made out of K copies of the SO$(2N)$ Lax matrix in such a way that each one contributes a factor of x^2 and therefore altogether we get x^{2K}. We note also that the first term in the above expression, by construction, is z-independent. So one can fix it with a proper value for z. Indeed setting $z = -i|\Lambda^{2KN-2K}|$ the first term can be written as

$$
\begin{aligned}
P_{2KN}(x) &= \prod_{t=1}^{K} \left[P_{2N}(x) - x^2 \eta \Lambda^{2N-2}\left(e^{-i\pi\frac{4t-1}{2K}} + e^{i\pi\frac{4t-1}{2K}}\right)\right] \\
&= 2^k x^{2K} \eta^K \Lambda^{2KN-2K} \prod_{t=1}^{K} \left[\frac{P_{2N}(x)}{2x^2\eta\Lambda^{2N-2}} - \cos\left(\pi \frac{4t-1}{2K}\right)\right] \\
&= 2x^K \eta^K \Lambda^{2NK-2K} T_K\left(\frac{P_{2N}(x)}{2\eta x^2 \Lambda^{2N-2}}\right),
\end{aligned} \tag{4.7}
$$

where $T_K(x)$ is the K-th Chebyshev polynomial of the first kind. Plugging the final result into the auxiliary spectral curve (4.6) we get

$$
x^{K-2}\left\{2x^2\eta^K \Lambda^{2NK-2K} T_K\left(\frac{P_{2N}(x)}{2\eta x^2 \Lambda^{2N-2}}\right) + x^2\left(z + \frac{\lambda^{4KN-4K}}{z}\right)\right\} = 0\,. \tag{4.8}
$$

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Since in general $x \neq 0$ we have

$$P_{2KN-2K+2}(x) + x^2\left(z + \frac{\lambda^{4KN-4K}}{z}\right) = 0 \tag{4.9}$$

with

$$P_{2KN-2K+2}(x) = 2x^2\eta^K\Lambda^{2KN-2K}T_K\left(\frac{P_{2N}(x)}{2\eta x^2\Lambda^{2N-2}}\right) \tag{4.10}$$

which is exactly the spectral curve for $\mathrm{SO}(2KN-2K+2)$ gauge group.

5. SYM theory with $\mathrm{SO}(2N+1)$ and $\mathrm{SP}(2N)$ gauge groups

5.1 $\mathrm{SO}(2N+1)$

In this section we shall briefly study $\mathcal{N}=2$ SYM theory with gauge group $\mathrm{SO}(2N+1)$ on $R^3\times S^1$. We note, however, that since the group is nonsimply laced one will have to work with its dual [18] where the short roots and long roots are exchanged. Therefore the integrable model one should deal with is a periodic Toda lattice based on the root system of dual gauge group [18] which for the case of B_N is given in terms of twisted Kac-Moody algebra $A^{(2)}_{2N-1}$.

The corresponding Lax matrix M can be expressed as the following

$$M = \sum_{i=1}^{N}(\phi_i\, h_i + y_i\, e_{\alpha_i} + e_{-\alpha_i}) + z\, e_{\alpha_0} + \frac{y_0}{z}\, e_{-\alpha_0}\,, \tag{5.1}$$

where h_i, e_{α_i} and e_{α_0} are generators of Cartan subalgebra, simple roots and affine root respectively. We note also that there is a constraint on y_i's given by $\prod_{j=0}^{N} y_j \prod_{i=1}^{N-2} y_i = \Lambda^{4N-2}$.

Using this Lax matrix the Seiberg-Witten curve for the four dimensional $\mathcal{N}=2$ $\mathrm{SO}(2N+1)$ SYM theory can be obtained from the spectral curve $\det(x\mathbf{1}-M)=0$ which is[4]

$$x(z+\frac{\Lambda^{4N-2}}{z}) + \frac{1}{2}(x^{2N}+u_2x^{2N-2}+\cdots+u_{2N}) = 0\,. \tag{5.2}$$

Let us now consider the theory on $R^3\times S^1$ with the following tree level superpotential

$$W = \frac{g_2}{2}\mathrm{Tr}\phi^2 + \frac{g_4}{4}\mathrm{Tr}\phi^4\,. \tag{5.3}$$

According to the proposal (3.5) the quantum superpotential is given by

$$W = \frac{g_2}{2}\mathrm{Tr}M^2 + \frac{g_4}{4}\mathrm{Tr}M^4 + 2L\log\left(\frac{\Lambda^{4N-2}}{\prod_{j=0}^{N} y_j\prod_{i=1}^{N-2} y_i}\right), \tag{5.4}$$

where M is the Lax matrix (5.1). Moreover we have also imposed the constraint on y_i using a Lagrange multiplier L. This superpotential is a function of ϕ_i, y_i and L.

[4]The Seiberg-Witten curve for gauge group $\mathrm{SO}(2N+1)$ was first proposed in [29].

To be specific let us consider SO(9) gauge group. By making use of the explicit form of the Lax matrix M and plugging it into the superpotential one can read the equations of motion of ϕ_i as the following

$$\begin{aligned}
W'(\phi_1)+g_4(2y_1\phi_1+y_0\phi_1+y_1\phi_2) &= 0\,,\\
W'(\phi_2)+g_4(y_1\phi_1+2y_1\phi_2+2y_2\phi_2+y_2\phi_3) &= 0\,,\\
W'(\phi_3)+g_4(y_2\phi_2+2y_2\phi_3+2y_3\phi_3+2y_4\phi_3+y_3\phi_4-y_4\phi_4) &= 0\,,\\
W'(\phi_4)+g_4(y_3\phi_3+2y_3\phi_4-y_4\phi_3+2y_4\phi_4) &= 0\,,
\end{aligned} \tag{5.5}$$

and for y_i's one finds

$$\begin{aligned}
g_2+g_4(2y_1+y_0+\phi_1^2) &= 2L\,y_0^{-1}\,,\\
g_2+g_4(y_2+y_1+y_0+\phi_1^2+\phi_2\phi_1+\phi_2^2) &= 2L\,y_1^{-1}\,,\\
g_2+g_4(y_1+y_2+y_3+y_4+\phi_2^2+\phi_2\phi_3+\phi_3^2) &= 2L\,y_2^{-1}\,,\\
g_2+g_4(3y_4+y_3+y_2+\phi_3^2+\phi_3\phi_4+\phi_4^2) &= L\,y_3^{-1}\,,\\
g_2+g_4(y_4+3y_3+y_2+\phi_3^2-\phi_3\phi_4+\phi_4^2) &= L\,y_4^{-1}\,,
\end{aligned} \tag{5.6}$$

finally for L we get $y_0y_1^2y_2^2y_3y_4=\Lambda^{14}$. These equations can be solved to find the minimum of the corresponding supersymmetric vacuum. For example consider a solution such that $\phi_i=0$ for $i=1,2,3,4$. Then the equations for y_i's can be solved leading to the following solution

$$y_0=y_1=y_2=2y\,,\qquad y_3=y_4=y\,,y=\frac{1}{2^{5/7}}\Lambda^2\,. \tag{5.7}$$

Plugging this solution into the superpotential one finds

$$W=7\left(2^{2/7}g_2\,\Lambda^2+2^{4/7}\frac{3}{2}\,g_4\,\Lambda^4\right), \tag{5.8}$$

which is exactly what we get from four dimensional field theory [31] if we rescale Λ as $\Lambda\to\frac{1}{2^{1/7}}\Lambda$. This can, for example, be seen from the Seiberg-Witten factorization (3.19). The same as that in the previous section this rescaling factor can also be understood from the spectral curve (5.2). In fact we have again an extra factor of 1/2 which can be absorbed in rescaling of z and Λ as $z\to z/2$ and $\Lambda\to\Lambda/2^{(2/4N-2)}$.

The supersymmetric vacua of SO$(2N+1)$ gauge theory with given superpotential can also be lifted to the supersymmetric vacua of SO$(2NK-K+1)$ with the same superpotential using the Chebyshev polynomials. This correspondence can also be seen using its corresponding integrable model. In fact the situation is the same as that in the previous section, of course with the Lax matrix of SO$(2N+1)$ gauge group. The final result is

$$2x\eta^K\Lambda^{2NK-K}T_K\left(\frac{P_{2N}(x)}{2\eta x^2\Lambda^{2N-1}}\right)+x\left(z+\frac{\lambda^{4KN-2K}}{z}\right)=0. \tag{5.9}$$

5.2 SP$(2N)$

Similarly one can also study the theory with gauge group SP$(2N)$. Being a nonsimply laced group points that we will have to work with its dual. In fact in was shown [18] that the

Seiberg-Witten curve for this theory can be obtained from a periodic Toda lattice based on the dual root system of $C_N^{(1)}$ which is given in terms of twisted Kac-Moody algebra $D_{N+1}^{(2)}$. Its Lax matrix has the same form as (5.1) but, of course, written in terms of $D_{N+1}^{(2)}$ root system. Moreover the constraint which should be imposed on y_i now becomes $\prod_{i=0}^{N} y_i^2 = \Lambda^{2N+2}$. Using this Lax matrix the spectral curve $\det(x\mathbf{1} - M) = 0$, reads

$$\left(z - \frac{\Lambda^{2N+2}}{z}\right)^2 + x^{2N} + u_2 x^{2N-2} + \cdots + u_{2n} = 0\,. \tag{5.10}$$

The same as in the previous section let us consider $\mathcal{N} = 2$ SP$(2N)$ SYM theory on $R^3 \times S^1$ deformed by a tree level superpotential given by (5.3). Following the proposal (3.5) the quantum superpotential reads

$$W = \frac{g_2}{2}\mathrm{Tr}M^2 + \frac{g_4}{4}\mathrm{Tr}M^4 + 2L\log\left(\frac{\Lambda^{2N+2}}{\prod_{i=0}^{N} y_i^2}\right). \tag{5.11}$$

Using the explicit form of Lax matrix we have evaluated the effective superpotential for the case of SP(6). In the case of maximally confining vacuum we have found

$$W = 8\left(g_2\Lambda^2 + \frac{3}{2}g_4\Lambda^4\right). \tag{5.12}$$

To compare this with the four-dimensional field theory we could, for example, study the factorization of the corresponding Seiberg-Witten curve given by [30]

$$y^2 = \left(x^2 P_{2N} + 2\Lambda^{2N+2}\right)^2 - 4\Lambda^{4N+4}\,. \tag{5.13}$$

In general the factorization corresponding to the situation where gauge group is broken as SP$(2N) \to$ SP$(2N_0) \times \prod_{i=1}^{n}$ U(N_i) is given by [31]

$$\left(x^2 P_{2N} + 2\Lambda^{2N+2}\right)^2 - 4\Lambda^{4N+4} = x^2 H_{2N-2n} F_{2(2n+1)}\,. \tag{5.14}$$

In the case of the maximally confining vacuum where $n = 0$ the solution for this problem is given in terms of Chebyshev polynomials (see also [32])

$$P_{2N}(x) = \frac{2\Lambda^{2N+2}}{x^2} T_{N+1}\left(\frac{x^2}{2\Lambda^2} - 1\right) - \frac{2\Lambda^{2N+2}}{x^2}\,. \tag{5.15}$$

Plugging this solution to the above hyperelliptic curve one finds

$$\begin{aligned} y^2 &= \left[2\Lambda^{2N+2} T_{N+1}\left(\frac{x^2}{2\Lambda^2} - 1\right)\right]^2 - 4\Lambda^{4N+4} \\ &= 4\Lambda^{4N+4}\left[T_{N+1}^2\left(\frac{x^2}{2\Lambda^2} - 1\right)^2 - 1\right] \\ &= x^2\Lambda^{4N}(x^2 - 4\Lambda^2) U_N^2\left(\frac{x^2}{2\Lambda^2} - 1\right). \end{aligned} \tag{5.16}$$

Form this solution one can read off the factorization points which can be found as the following

$$T_{N+1}\left(\frac{x_r^2}{2\Lambda^2}-1\right)=0\,,\quad\rightarrow\quad x_r^2=(2\Lambda)^2\cos^2\left(\frac{(r-\frac{1}{2})\pi}{2(N+1)}\right),\tag{5.17}$$

and therefore the gauge invariant parameters of the theory are given by

$$\langle u_{2p}\rangle=\frac{N+1}{p}\binom{2p}{p}\Lambda^{2p}\,.\tag{5.18}$$

By making use of this solution the exact superpotential reads

$$W=g_2\langle u_2\rangle+g_4\langle u_4\rangle=8\left(g_2\Lambda^2+\frac{3}{2}g_4\Lambda^4\right),\tag{5.19}$$

in agreement with (5.12).

Finally we note that the lifting from SP($2N$) to SP($2N+2K-2$) gauge theory can also be done using the corresponding Lax matrices.

6. Conclusions

In this paper we have studied deformed $\mathcal{N}=2$ supersymmetric gauge theory with classical gauge group on $R^3\times S^1$. The deformation is given by a tree level superpotential. We have seen that the effective quantum superpotential can be obtained by making use of an integrable model which is related to the four dimensional theory. This integrable model is the periodic Toda lattice which is based on the root lattice of the affine Lie algebra corresponding to the gauge group. We note, however, that for nonsimply laced gauge group we need to use its dual which turns out to be twisted Kac-Moody algebra [33]. This is, of course, what we expected from the relation between periodic Toda lattice and Seiberg-Witten models [18].

The prescription to find the effective superpotential is simple and it just needs a substitution. More precisely if the tree level superpotential is Tr$W(\phi)$ then the effective superpotential is given by Tr$W(M)$ where M is the Lax matrix of the corresponding integrable model. As we see we do not even need performing any integrations. This might be understood from the point that a $(4+d)$-dimensional gauge theory could be studied by a d-dimensional auxiliary theory [34]. Therefore in a three dimensional theory it seems that no integrations are indeed needed.

We note also that since the structure of the supersymmetric vacua on $R^3\times S^1$ is R independent one could compare our results with that on R^4 theory. In fact we have shown that it precisely gives the same vacuum structure of R^4 theory.

In our expressions for the effective superpotential a Lagrange multiplier is involved which has been used to impose the constraint on the parameters of the integrable model. Following [10] this Lagrange multiplier can be identified with the glueball field S. Although in the case where classically the gauge group remains unbroken one can integrate in the glueball field S, for the situation where the gauge group is classically broken this procedure remains unsolved. Moreover even in the first case it would be quite interesting to see how one can go from $R^3\times S^1$ to R^4 getting for example the Veneziano-Yankielowicz superpotential.

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